Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 259 (2010) 2437–2454
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

The dual space of $(\mathcal{L}(X,Y),\tau_p)$ and the p-approximation property [☆]

Yun Sung Choi [a], Ju Myung Kim [b,*]

[a] *Department of Mathematics, POSTECH, San 31 Hyoja Dong, Nam-Gu, Pohang, Gyeong-Buk, 790-784, Republic of Korea*

[b] *Pohang Mathematics Institute, POSTECH, San 31 Hyoja Dong, Nam-Gu, Pohang, Gyeong-Buk, 790-784, Republic of Korea*

Received 4 April 2010; accepted 23 June 2010

Communicated by N. Kalton

Abstract

We establish a representation of the dual space of $\mathcal{L}(X,Y)$, the space of bounded linear operators from a Banach space X into a Banach space Y, endowed with the topology τ_p of uniform convergence on p-compact subsets of X. We apply this representation and solve the duality problem for the p-approximation property (p-AP), that is, if the dual space X^* has the p-AP, then so does X. However, the converse does not hold in general. We show that given $2 < p < \infty$, there exists a subspace of l_q which fails to have the p-AP, when $q > 2p/(p-2)$. This subspace is the Davie space in l_q (Davie (1973) [5]) which does not have the approximation property. It follows that for every $2 < p < \infty$ there exists a Banach space Y_p such that it has the p-AP, but its dual space Y_p^* fails to have the p-AP. We study the relation of the p-AP with the denseness of finite rank operators in the topology τ_p. Finally we introduce the p-compact approximation property (p-CAP) and show for every $2 < p < \infty$ that the Davie space in c_0 fails to have the p-CAP, and also that a variant of the Willis space (Willis (1992) [17]) has the p-CAP, but it fails to have the p-AP.

Keywords: p-Compact set; p-Compact operator; p-Approximation property; p-Compact approximation property

☆ This work was supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0094068).

* Corresponding author.
E-mail addresses: mathchoi@postech.ac.kr (Y.S. Choi), kjm21@kaist.ac.kr (J.M. Kim).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.06.017

1. Introduction

The approximation property, which was systematically investigated by Grothendieck [9], is one of the most important properties in Banach space theory. A Banach space X is said to have the *approximation property* (AP) if for every compact set K in X and every $\varepsilon > 0$, there exists a finite rank operator T on X such that $\sup_{x\in K} \|Tx - x\| \leqslant \varepsilon$. He characterized relatively compact sets in Banach spaces in such a way that a subset K of X is relatively compact if and only if there exists a null sequence (x_n) in X such that $K \subset \overline{abco}(\{x_n\}) = \{\sum_n \alpha_n x_n\colon (\alpha_n) \in B_{l_1}\}$, where $\overline{abco}(\{x_n\})$ is the absolutely convex hull of $\{x_n\}$ and B_X is the closed unit ball of a Banach space X (see [12, Proposition 1.e.2]).

Motivated by this characterization, Sinha and Karn [15,16] introduced and studied p-compacts sets, p-compact operators and the p-approximation property (for short, p-AP) in the following sense.

For $1 \leqslant p \leqslant \infty$, a subset K of X is said to be *relatively p-compact* if $K \subset \{\sum_n \alpha_n x_n\colon (\alpha_n) \in B_{l_{p^*}}\}$, where $\frac{1}{p} + \frac{1}{p^*} = 1$ and $(x_n) \in l_p(X)$ $(1 \leqslant p < \infty)$ $((x_n) \in c_0(X) \subset l_\infty(X)$ if $p = \infty)$, where $l_p(X)$ and $c_0(X)$ are the Banach spaces of X-valued p-summable and null sequences, respectively. Note that the ∞-compact sets are precisely the compact sets and every p-compact set is q-compact for $1 \leqslant p < q \leqslant \infty$. We can also see that the closed convex hull $\overline{co}(K)$ of a relatively p-compact set K is also p-compact, because the set $\{\sum_n \alpha_n x_n\colon (\alpha_n) \in B_{l_{p^*}}\}$ is closed and convex.

The concept of a p-compact set leads naturally to that of the p-AP, which can be considered as a way to weaken the approximation property. For $1 \leqslant p \leqslant \infty$, a Banach space X is said to have the p-AP if for every p-compact subset K of X and every $\varepsilon > 0$ there exists a finite rank operator T on X such that $\sup_{x\in K} \|Tx - x\| \leqslant \varepsilon$. In fact, the ∞-AP means the AP. We can see easily that if X has the q-AP, then X has the p-AP for $1 \leqslant p < q \leqslant \infty$. An interesting result on the p-AP is that every Banach space has the 2-AP [15, Theorem 6.4] (hence the p-AP for every $1 \leqslant p \leqslant 2$). In case of the AP, Enflo [8] showed that there is a separable Banach space which does not have the AP.

Grothendieck [9] initiated the study of the variants of the AP and relations between them. The important tools he used were the topology τ on $\mathcal{L}(X, Y)$ of uniform convergence on compact sets in X and the representation of the continuous linear functionals on $(\mathcal{L}(X, Y), \tau)$. They were applied to show the relation of the AP with the denseness of finite rank operators, and also such a basic property of the AP that if X^* has the AP, then so does X. However, it was not solved that if X^* has the p-AP, then so does X, even though Delgado et al. [6] showed that if X^{**} has the p-AP, then so does X. Our main interest in this paper is to find the representation of the continuous linear functionals on $(\mathcal{L}(X, Y), \tau_p)$, where τ_p is the topology on $\mathcal{L}(X, Y)$ of uniform convergence on p-compact sets in X. We apply it to solve the above duality problem for the p-AP and study the relation of the p-AP with the denseness of finite rank operators in the topology τ_p. In general, even though X has the p-AP, X^* does not necessarily have the p-AP. Indeed, we show that given $2 < p < \infty$, there exists a subspace B_q of l_q for every $q > 2p/(p-2)$ such that B_q does not have the p-AP. This space B_q is the Davie space in l_q [5] which does not have the AP. It follows that for every $2 < p < \infty$ there exists a Banach space Y_p such that it has the p-AP, but its dual space Y_p^* fails to have the p-AP.

Let $\mathcal{F}(X, Y)$ (resp. $\mathcal{K}(X, Y)$) be the space of finite rank (resp. compact) operators from X into Y. The concept of a p-compact operator was defined in [15] in the following manner. For $1 \leqslant p \leqslant \infty$, an operator $T : X \to Y$ is said to be *p-compact* if $T(B_X)$ is relatively p-compact in Y. Let $\mathcal{K}_p(X, Y)$ be the space of p-compact operators from X into Y.

Sinha and Karn [15] showed that an operator $T \in \mathcal{L}(X,Y)$ is p-compact if and only if it is quotiented in l_{p^*}. In fact, they showed that $T \in \mathcal{K}_p(X,Y)$ factors into the composition of a p-compact operator and a weakly p-compact operator (see [15] for its definition) through a quotient space of l_{p^*}. We show that a p-compact operator factors into the composition of a compact operator and a p-compact operator through a quotient space of l_1. This factorization is also applied to study the relation of the p-AP with the denseness of finite rank operators.

It is an immediate result from the definition of the p-AP that (a) X has the p-AP $\Leftrightarrow$ (b) $\overline{\mathcal{F}(Y,X)}^{\tau_p} = \mathcal{L}(Y,X)$ for every Banach space Y $\Leftrightarrow$ (c) $\overline{\mathcal{F}(X,Y)}^{\tau_p} = \mathcal{L}(X,Y)$ for every Banach space Y (see [15]). This result was improved by Delgado et al. [6] in such a way that (a) X has the p-AP $\Leftrightarrow$ (b) $\mathcal{K}_p(Y,X) \subset \overline{\mathcal{F}(Y,X)}^{\|\cdot\|}$ for every Banach space Y $\Leftrightarrow$ (c) $\mathcal{K}_p(Y,X) \subset \overline{\mathcal{F}(Y,X)}^{\tau}$ for every Banach space Y.

In case of the AP we recall that X has the AP $\Leftrightarrow$ $\overline{\mathcal{F}(Y,X)}^{\tau} = \mathcal{L}(Y,X)$ for every Banach space Y $\Leftrightarrow$ $\overline{\mathcal{F}(X,Y)}^{\tau} = \mathcal{L}(X,Y)$ for every Banach space Y $\Leftrightarrow$ $\overline{\mathcal{F}(Y,X)}^{\|\cdot\|} = \mathcal{K}(Y,X)$ for every Banach space Y (see [2, Theorem 2.5]).

Further, it was shown in [4,10] that X has the AP $\Leftrightarrow$ $\mathcal{K}(Y,X) \subset \overline{\mathcal{F}(Y,X)}^{\tau}$ for every Banach space Y $\Leftrightarrow$ $\mathcal{K}(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\tau}$ for every Banach space Y.

From this point of view we show that X has the p-AP $\Leftrightarrow$ $\mathcal{K}(Y,X) \subset \overline{\mathcal{F}(Y,X)}^{\tau_p}$ for every Banach space Y $\Leftrightarrow$ $\mathcal{K}(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\tau_p}$ for every Banach space Y. Compared with the above result by Delgado et al., we note that $\tau_p \leqslant \tau$ and $\mathcal{K}_p(X,Y) \subset \mathcal{K}(X,Y)$.

In the last part of the paper we introduce the p-compact approximation property (p-CAP) and show for every $2 < p < \infty$ that the Davie space in c_0 fails to have the p-CAP, and also that a variant of the Willis space [17] has the p-CAP, but it fails to have the p-AP.

2. The Dual space $(\mathcal{L}(X,Y), \tau_p)^*$ and the p-AP

Let $1 \leqslant p \leqslant \infty$. For a p-compact subset K of X, $\delta > 0$, and $T \in \mathcal{L}(X,Y)$, we put

$$\mathcal{N}_p(T;K,\delta) = \Big\{ R \in \mathcal{L}(X,Y)\colon \sup_{x\in K} \|Rx - Tx\| < \delta \Big\}.$$

Then the collection of all such sets $\mathcal{N}_p(T;K,\delta)$'s forms a topological basis on $\mathcal{L}(X,Y)$ and let τ_p be the topology generated by this basis, which is the topology on $\mathcal{L}(X,Y)$ of uniform convergence on p-compact sets in X introduced by Sinha and Karn [15]. We observe that the topology τ_p is a completely regular and locally convex. For a net (T_α) and T in $\mathcal{L}(X,Y)$ it is easily seen that

$$T_\alpha \xrightarrow{\tau_p} T \quad \text{if and only if} \quad \sup_{x\in K} \|T_\alpha x - Tx\| \to 0$$

for every p-compact subset K of X, and also that for $\mathcal{A} \subset \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(X,Y)$, $T \in \overline{\mathcal{A}}^{\tau_p}$ if and only if for every p-compact subset K of X and every $\varepsilon > 0$ there exists $S \in \mathcal{A}$ such that $\sup_{x\in K} \|Sx - Tx\| \leqslant \varepsilon$. Thus the *topology τ of uniform convergence on compact sets* is stronger than the topology τ_p for every $1 \leqslant p < \infty$, and τ_q is also stronger than τ_p for $1 \leqslant p < q < \infty$. We can also find that X has the AP if and only if $id_X \in \overline{\mathcal{F}(X,X)}^{\tau}$, and that X has the p-AP if and only if $id_X \in \overline{\mathcal{F}(X,X)}^{\tau_p}$, where id_X is the identity operator on X.

We now show that if X is infinite-dimensional, then the topology τ_p on $\mathcal{L}(X,Y)$ is not metrizable.

Proposition 2.1. *Let $1 \leqslant p \leqslant \infty$. Then the topology τ_p on $\mathcal{L}(X, Y)$ is metrizable if and only if X is finite-dimensional.*

Proof. The proof follows from the same argument as in the proof of [3, Theorems 2.7 and 3.8]. $\square$

It was shown that $(\mathcal{L}(X, Y), \tau)$ is complete [3, Theorem 3.10(a)], and so is $(\mathcal{L}(X, Y), \tau_p)$.

Proposition 2.2. *Let $1 \leqslant p \leqslant \infty$. Then $(\mathcal{L}(X, Y), \tau_p)$ is complete.*

Proof. Suppose that (T_α) is a τ_p-Cauchy net in $\mathcal{L}(X, Y)$. Then $(T_\alpha x)$ is also a Cauchy net in Y for every $x \in X$. Define an operator $T : X \to Y$ by $Tx = \lim_\alpha T_\alpha x$. Clearly T is linear. If T is unbounded, then there exists a sequence (x_n) in X such that $\|x_n\| < 1/n^2$ and $\|Tx_n\| > n$ for every n. Since $(\sum_n \|x_n\|^p)^{1/p} \leqslant \sum_n \|x_n\| \leqslant \sum_n 1/n^2 < \infty$, we have $(x_n) \in l_p(X)$. Hence the set $\{x_n\}$ is relatively p-compact, which implies

$$\lim_{\alpha,\beta} \sup_n \big\| (T_\alpha - T_\beta)x_n \big\| = 0.$$

It follows that there exists α_0 such that $\sup_n \|(T_\alpha - T_\beta)x_n\| \leqslant 1$ for all $\alpha, \beta \succcurlyeq \alpha_0$. Since $\lim_\alpha T_\alpha x = Tx$ for every $x \in X$, $\sup_n \|(T_{\alpha_0} - T)x_n\| \leqslant 1$. Thus for every n

$$\|T_{\alpha_0}\| \geqslant \|T_{\alpha_0} x_n\| \geqslant \|Tx_n\| - \big\|(T_{\alpha_0} - T)x_n\big\| > n - 1,$$

which contradicts that T_{α_0} is bounded. Therefore T is bounded. By the similar argument to the above we can show $T_\alpha \xrightarrow{\tau_p} T$, hence $(\mathcal{L}(X, Y), \tau_p)$ is complete. $\square$

The following is the τ_p-version of Mazur's theorem. Recall that the *strong operator topology* τ_{sto} on $\mathcal{L}(X, Y)$ is the topology of uniform convergence on finite subsets of X.

Proposition 2.3. *Let $1 \leqslant p < \infty$. If $\mathcal{C}$ is a τ_p-compact set in $\mathcal{L}(X, Y)$, then $\overline{co}^{\tau_p}(\mathcal{C})$ is τ_p-compact.*

Proof. If $\mathcal{C}$ is τ_p-compact, then $\mathcal{C}$ is compact in the strong operator topology τ_{sto}. By [3, Theorem 3.13], $\overline{co}^{\tau_{sto}}(\mathcal{C}) = \overline{co}^{\tau}(\mathcal{C})$ is τ-compact and so is τ_p-compact. Since $\overline{co}^{\tau}(\mathcal{C}) \subset \overline{co}^{\tau_p}(\mathcal{C}) \subset \overline{co}^{\tau_{sto}}(\mathcal{C}) = \overline{co}^{\tau}(\mathcal{C})$, we have the conclusion. $\square$

We now establish a representation of $(\mathcal{L}(X, Y), \tau_p)^*$, which is applied to solve the duality problem for the p-AP.

Let $l_p^w(X)$ be the Banach space of X-valued weakly p-summable sequences with the norm $\|(x_n)\|_p^w = \sup_{x^* \in B_{X^*}} (\sum_n |x^*(x_n)|^p)^{1/p}$. Let $\check{l}_p^w(X)$ be a subspace of $l_p^w(X)$ such that

$$\sup_{x^* \in B_{X^*}} \left(\sum_{n \geqslant m} \big|x^*(x_n)\big|^p \right)^{1/p} \to 0 \quad \text{as } m \to \infty$$

for $(x_n) \in \check{l}_p^w(X)$.

Lemma 2.4. *Let $1 < p < \infty$. Then $\check{l}_p^w(X)^*$ consists of all linear functionals φ of the form*

$$\varphi((x_n)) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j x_j^*(x_n),$$

where $z_j = (\lambda_n^j)_{n=1}^{\infty}$ in l_{p^*} for each $j \in \mathbb{N}$ and (x_j^*) in X^* with

$$\sum_{j=1}^{\infty} \|z_j\|_{p^*} \|x_j^*\| < \infty.$$

Proof. If φ is of the above form $\varphi((x_n)) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j x_j^*(x_n)$, then we see that $\varphi \in \check{l}_p^w(X)^*$ and $\|\varphi\| \leqslant \sum_{j=1}^{\infty} \|(\lambda_n^j)\|_{p^*} \|x_j^*\|$.

Recall that the space $\check{l}_p^w(X)$ is isometrically isomorphic to the injective tensor product $l_p \check{\otimes} X$, that is, every element (x_n) in $\check{l}_p^w(X)$ is identified with the element $\sum_n e_n \otimes x_n$ in $l_p \check{\otimes} X$ (see [7, Corollary 1.1.12]), and $(l_p \check{\otimes} X)^*$ is isometrically isomorphic to the projective tensor product $l_{p^*} \hat{\otimes} X^*$ (see [14, Theorem 5.33]). Consequently, $\check{l}_p^w(X)^*$ is isometrically isomorphic to $l_{p^*} \hat{\otimes} X^*$.

Now suppose that $\varphi \in \check{l}_p^w(X)^*$. Then there exist $z_j = (\lambda_n^j)_{n=1}^{\infty}$ in l_{p^*} for each $j \in \mathbb{N}$ and $(x_j^*)_{j=1}^{\infty}$ in X^* with $\sum_{j=1}^{\infty} \|z_j\|_{p^*} \|x_j^*\| < \infty$ such that for every $(x_n) \in \check{l}_p^w(X)$

$$\varphi((x_n)) = \varphi\left(\sum_n e_n \otimes x_n\right) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \left[(z_j)e_n\right] x_j^*(x_n) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j x_j^*(x_n). \qquad \square$$

We are now ready to obtain a representation of $(\mathcal{L}(X,Y), \tau_p)^*$. Recall that a linear functional f on a topological vector space $\mathcal{V}$ is continuous if and only if there exists a neighborhood U of 0 in $\mathcal{V}$ such that $f(U)$ is bounded (see [13, Theorem 2.2.16]). Thus $f \in (\mathcal{L}(X,Y), \tau_p)^*$ if and only if there exist $M > 0$ and a p-compact subset K of X such that $|f(T)| \leqslant M \sup_{x\in K} \|Tx\|$ for every $T \in \mathcal{L}(X,Y)$. The proof of the following theorem is based on the argument in the representation of the dual space $(\mathcal{L}(X,Y), \tau)^*$ given by Grothendieck [9] (see [2, Proposition 2.4] or [12, Proposition 1.e.3]).

Theorem 2.5. *Let $1 < p < \infty$. Then $(\mathcal{L}(X,Y), \tau_p)^*$ consists of all linear functionals f of the form*

$$f(T) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j y_j^*(Tx_n),$$

where $(x_n) \in l_p(X)$, $z_j = (\lambda_n^j)_{n=1}^{\infty}$ in l_{p^*} for each $j \in \mathbb{N}$ and (y_j^*) in Y^* with $\sum_{j=1}^{\infty} \|z_j\|_{p^*} \|y_j^*\| < \infty$.

Proof. Suppose that f is of the above form

$$f(T)=\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\lambda_n^j y_j^*(Tx_n).$$

Lemma 2.4 shows that

$$\varphi:=\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\lambda_n^j y_j^*(\cdot)\in \check{l}_p^w(Y)^*.$$

Since $(Tx_n)_{n=1}^{\infty}\in \check{l}_p^w(Y)$ for every $T\in\mathcal{L}(X,Y)$, we see that $f(T)=\varphi((Tx_n))$ for every $T\in\mathcal{L}(X,Y)$. Consider the p-compact set $K=\{\sum_n \alpha_n x_n\colon (\alpha_n)\in B_{l_{p^*}}\}$ in X. It is easy to check that $\|(Tx_n)\|_p^w=\sup_{x\in K}\|Tx\|$ for every $T\in\mathcal{L}(X,Y)$. For every $T\in\mathcal{N}_p(0;K,1)$,

$$\left|f(T)\right|=\left|\varphi((Tx_n))\right|\leqslant\|\varphi\|\left\|(Tx_n)\right\|_p^w=\|\varphi\|\sup_{x\in K}\|Tx\|\leqslant\|\varphi\|,$$

hence $f\in(\mathcal{L}(X,Y),\tau_p)^*$.

Conversely, suppose $f\in(\mathcal{L}(X,Y),\tau_p)^*$. Then there exist $M>0$ and a p-compact set K in X such that $|f(T)|\leqslant M\sup_{x\in K}\|Tx\|$ for every $T\in\mathcal{L}(X,Y)$. Let $(x_n)\in l_p(X)$ such that

$$K\subset\left\{\sum_n \alpha_n x_n\colon (\alpha_n)\in B_{l_{p^*}}\right\}.$$

Then $|f(T)|\leqslant M\|(Tx_n)\|_p^w$ for every $T\in\mathcal{L}(X,Y)$. Consider the subspace $\{(Tx_n)\colon T\in\mathcal{L}(X,Y)\}$ of $\check{l}_p^w(Y)$ and the functional φ on $\{(Tx_n)\colon T\in\mathcal{L}(X,Y)\}$ given by $\varphi((Tx_n))=f(T)$. If $(Tx_n)=(Rx_n)$, then $f(T)=f(R)$, because $|f(T-R)|\leqslant M\|((T-R)x_n)\|_p^w$. Thus φ is well defined and linear. Since

$$\left|\varphi((Tx_n))\right|=\left|f(T)\right|\leqslant M\left\|(Tx_n)\right\|_p^w,$$

φ is a bounded linear functional on the subspace $\{(Tx_n)\colon T\in\mathcal{L}(X,Y)\}$ of $\check{l}_p^w(Y)$. Then there exists a Hahn–Banach extension $\hat{\varphi}\in\check{l}_p^w(Y)^*$ of φ such that $f(T)=\varphi((Tx_n))=\hat{\varphi}((Tx_n))$ for every $T\in\mathcal{L}(X,Y)$. By Lemma 2.4 there exist $z_j=(\lambda_n^j)_{n=1}^{\infty}$ in l_{p^*} for each $j\in\mathbb{N}$ and (y_j^*) in Y^* with $\sum_{j=1}^{\infty}\|z_j\|_{p^*}\|y_j^*\|<\infty$ such that $\hat{\varphi}((y_n))=\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\lambda_n^j y_j^*(y_n)$ for every $(y_n)\in\check{l}_p^w(Y)$. Therefore we conclude that

$$f(T)=\hat{\varphi}((Tx_n))=\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\lambda_n^j y_j^*(Tx_n)$$

for every $T\in\mathcal{L}(X,Y)$. □

We now apply this representation and solve the duality problem for the p-AP, that is, if X^* has the p-AP, the so does X.

Lemma 2.6. *(See [12, Lemma 1.e.17].) For every Banach space X, $\mathcal{F}(X^*, X^*) \subset \overline{\mathcal{F}^*(X, X)}^{\tau}$, where $\mathcal{F}^*(X, X) = \{T^*\colon T \in \mathcal{F}(X, X)\}$.*

Consequently, X^* has the p-AP if and only if $id_{X^*} \in \overline{\mathcal{F}^*(X, X)}^{\tau_p}$.

Theorem 2.7. *Let $2 < p < \infty$. If X^* has the p-AP, then X has the p-AP.*

Proof. It is enough to show that if $f \in (\mathcal{L}(X, X), \tau_p)^*$ and $f(T) = 0$ for all $T \in \mathcal{F}(X, X)$, then $f(id_X) = 0$.

By Theorem 2.5 there exist $(x_n) \in l_p(X)$, $z_j = (\lambda_n^j)_{n=1}^\infty$ in l_{p^*} and (x_j^*) in X^* with $\sum_{j=1}^\infty \|z_j\|_{p^*}\|x_j^*\| < \infty$ such that

$$f(T) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j^*(T x_n)$$

for every $T \in \mathcal{L}(X, X)$. We may assume without loss of generality that $\|z_j\|_{p^*} \leqslant 1$ for every j and $\sum_{j=1}^\infty \|x_j^*\| < \infty$.

On the other hand, we can write

$$f(T) = \sum_{n=1}^\infty \sum_{j=1}^\infty \bigl(\lambda_n^j \|x_j^*\|^{\frac{1}{p^*}}\bigr) j_X(x_n) T^*\bigl(\|x_j^*\|^{-\frac{1}{p^*}} x_j^*\bigr),$$

where $j_X : X \to X^{**}$ is the canonical isometry. Define

$$g(S) = \sum_{n=1}^\infty \sum_{j=1}^\infty \bigl(\lambda_n^j \|x_j^*\|^{\frac{1}{p^*}}\bigr) j_X(x_n) S\bigl(\|x_j^*\|^{-\frac{1}{p^*}} x_j^*\bigr), \quad S \in \mathcal{L}\bigl(X^*, X^*\bigr).$$

Then an easy computation shows that

$$\bigl(\|x_j^*\|^{-\frac{1}{p^*}} x_j^*\bigr)_{j=1}^\infty \in l_p\bigl(X^*\bigr)$$

and

$$\bigl(\lambda_n^j \|x_j^*\|^{\frac{1}{p^*}}\bigr)_{j=1}^\infty \in l_{p^*}$$

for every n. Moreover,

$$\begin{aligned}
&\sum_{n=1}^\infty \bigl\| \bigl(\lambda_n^j \|x_j^*\|^{\frac{1}{p^*}}\bigr)_{j=1}^\infty \bigr\|_{p^*} \bigl\| j_X(x_n) \bigr\| \\
&\quad \leqslant \left(\sum_{n=1}^\infty \|x_n\|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \sum_{j=1}^\infty |\lambda_n^j|^{p^*} \|x_j^*\| \right)^{\frac{1}{p^*}}
\end{aligned}$$

$$= \left(\sum_{n=1}^{\infty}\|x_n\|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\left|\lambda_n^j\right|^{p^*}\left\|x_j^*\right\|\right)^{\frac{1}{p^*}}$$

$$\leqslant \left(\sum_{n=1}^{\infty}\|x_n\|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{\infty}\left\|x_j^*\right\|\right)^{\frac{1}{p^*}} < \infty.$$

Thus it follows from Theorem 2.5 that $g \in (\mathcal{L}(X^*, X^*), \tau_p)^*$.

By the assumption, $f(T) = 0$ for all $T \in \mathcal{F}(X, X) \Leftrightarrow g(T^*) = 0$ for all $T \in \mathcal{F}(X, X)$. Since X^* has the p-AP, we have $\overline{\mathcal{F}(X^*, X^*)}^{\tau_p} = \mathcal{L}(X^*, X^*)$, hence $\overline{\mathcal{F}^*(X, X)}^{\tau_p} = \mathcal{L}(X^*, X^*)$ by Lemma 2.6. This implies that $g(id_{X^*}) = 0$, hence $f(id_X) = g(id_{X^*}) = 0$, which completes the proof. □

However, the converse of Theorem 2.7 is not true in general. We show that for every $2 < p < \infty$ there exists a Banach space Y_p such that it has the p-AP, but its dual space Y_p^* fails to have the p-AP. We start with the following variant of the Davie space [5] which does not have the AP.

Let $G_0 = \{1, 2, 3 \cdot 2^0\}$ and for each positive integer k,

$$G_k = \left\{\max G_{k-1} + 1, \max G_{k-1} + 2, \ldots, \max G_{k-1} + 3 \cdot 2^k\right\}.$$

Then $|G_k| = 3 \cdot 2^k$ for $k = 0, 1, 2, \ldots,$ $G_k \cap G_l = \phi$ for each $k \neq l$, and $\mathbb{N} = \bigcup_{k=0}^{\infty} G_k$.

Let $1 \leqslant q, r < \infty$. We consider the Banach space

$$Z_{q,r} = \left\{f\colon \mathbb{N} \to \mathbb{C} \,\Big|\, \sum_{k=0}^{\infty}\left(\sum_{g\in G_k}\left|f(g)\right|^q\right)^{\frac{r}{q}} < \infty\right\},$$

with the norm

$$\|f\|_{Z_{q,r}} = \left(\sum_{k=0}^{\infty}\left(\sum_{g\in G_k}\left|f(g)\right|^q\right)^{\frac{r}{q}}\right)^{\frac{1}{r}},$$

which is isometrically isomorphic to $(\sum_{k=0}^{\infty} l_q^{3\cdot 2^k})_r$. Note $Z_{q,q} = l_q$.

We use the same notations in [5]. For each $k \geqslant 0$ and $1 \leqslant j \leqslant 2^k$, let $e_j^k : \mathbb{N} \to \mathbb{C}$ be the function in [5]. Let $B_{q,r}$ be the closed linear span of $\{e_j^k\colon k \geqslant 0,\ j = 1, \ldots, 2^k\}$ in $Z_{q,r}$. For each $k \geqslant 0$, let β^k be the linear functional on $\mathcal{L}(B_{q,r}, B_{q,r})$ and $\phi_g^k : \mathbb{N} \to \mathbb{C}$ the function for each $g \in G_k$ in [5]. Then as in the proof of [5], we can show that

(2.1) there exists $A_{q,r} > 0$ such that for every $k \geqslant 0$

$$\left\|\phi_g^k\right\|_{Z_{q,r}} \leqslant A_{q,r}(k+1)^{\frac{1}{2}} 2^{-\frac{k}{2}+\frac{k}{q}}$$

for all $g \in G_k$,

(2.2) for every $T \in \mathcal{L}(B_{q,r}, B_{q,r})$

$$\lim_k \beta^k(T) = \beta(T) \text{ exists}, \qquad \left|\beta(T)\right| \leqslant 3 \sup_n \|Tx_n\|_{Z_{q,r}}, \quad \text{and} \quad \beta(id_{B_{q,r}}) = 1,$$

where

$$(x_n) = \big(e_1^0, \phi_{g_1}^0, \phi_{g_2}^0, \phi_{g_3}^0, (1+1)^2\phi_{g_1}^1, \ldots, (1+1)^2\phi_{g_{3\cdot 2}}^1, \\ \ldots, (k+1)^2\phi_{g_1}^k, \ldots, (k+1)^2\phi_{g_{3\cdot 2^k}}^k, \ldots\big).$$

Theorem 2.8. *Let $1 \leqslant r < \infty$ and $2 < p < \infty$. Then $B_{q,r}$ does not have the p-AP, when $q > 2p/(p-2)$.*

Proof. Let $q > 2p/(p-2)$. It follows from (2.1) that

$$\begin{aligned}\sum_n \|x_n\|_{Z_{q,r}}^p &= \|e_1^0\|_{Z_{q,r}}^p + \sum_{k=0}^{\infty} \sum_{g \in G_k} \|(k+1)^2 \phi_g^k\|_{Z_{q,r}}^p \\ &\leqslant \|e_1^0\|_{Z_{q,r}}^p + A_{q,r} \sum_{k=0}^{\infty} 3 \cdot 2^k (k+1)^{5p/2} 2^{-\frac{pk}{2} + \frac{pk}{q}} \\ &= \|e_1^0\|_{Z_{q,r}}^p + 3A_{q,r} \sum_{k=0}^{\infty} (k+1)^{5p/2} 2^{\frac{(2p+2q-pq)k}{2q}} < \infty,\end{aligned}$$

hence $(x_n) \in l_p(B_{q,r})$. We also deduce from (2.2) that

$$\beta \in \big(\mathcal{L}(B_{q,r}, B_{q,r}), \tau_p\big)^*.$$

The proof of [5] and (2.2) show that

$$\beta(id_{B_{q,r}}) = 1 \quad \text{and} \quad \beta(T) = 0$$

for every $T \in \mathcal{F}(B_{q,r}, B_{q,r})$, hence $id_{B_{q,r}} \notin \overline{\mathcal{F}(B_{q,r}, B_{q,r})}^{\tau_p}$. Therefore $B_{q,r}$ does not have the p-AP. $\square$

Corollary 2.9. *Let $2 < p < \infty$. For every $2p/(p-2) < q < \infty$, the Banach space $B_{q,q}$ is a subspace of l_q failing to have the p-AP.*

Lemma 2.10. *(See [2, Proposition 1.3].) If X is a separable Banach space, then there exists a separable Banach space Y such that Y^{**} has a basis and Y^{***} is isomorphic to $Y^* \oplus X^*$.*

The following theorem follows from Corollary 2.9 and Lemma 2.10.

Theorem 2.11. *For every $2 < p < \infty$ there exists a separable Banach space X_p such that X_p^{**} has a basis, but X_p^{***} fails to have the p-AP.*

3. A factorization of p-compact operators

Sinha and Karn [15] showed that an operator is p-compact if and only if it is quotiented in l_{p^*}. More precisely, for $T \in \mathcal{K}_p(X, Y)$ there is a $y = (y_n) \in l_p(Y)$ such that

$$T(B_X) \subset \left\{ \sum_n \alpha_n y_n \colon (\alpha_n) \in B_{l_{p^*}} \right\}.$$

Define a bounded operator $T_y : l_{p^*} \to Y$ by $T_y(\alpha) = \sum_{n=1}^{\infty} \alpha_n y_n$, $\alpha = (\alpha_n) \in l_{p^*}$. Define $\hat{T}_y : l_{p^*}/ker(T_y) \to Y$ by $\hat{T}_y([\alpha]) = T_y(\alpha)$. For each $x \in X$, there exists $\beta = (\beta_n) \in l_{p^*}$ such that $Tx = \sum_{n=1}^{\infty} \beta_n y_n$. Define a bounded operator $Q_y : X \to l_{p^*}/ker(T_y)$ by $Q_y(x) = [\beta]$. Then we can see easily that $T = \hat{T}_y Q_y$. Here, we notice that $\hat{T}_y$ is p-compact and Q_y is weakly p-compact.

We show that the operator T_y factors further into $V \circ U$ through l_1, where U is p-compact and V is compact, hence a p-compact operator T factors into the composition of a compact operator and a p-compact operator through a quotient space of l_1. This factorization is also applied to study the relation of the p-AP with the denseness of finite rank operators. The same notations as in the above are used in the following.

Theorem 3.1. *Let $1 \leqslant p < \infty$. Then $T : X \to Y$ is p-compact if and only if there exist $y \in l_p(Y)$, a closed subspace M of l_1, a p-compact operator $\hat{U} : l_{p^*}/ker(T_y) \to l_1/M$, and a compact operator $\hat{V} : l_1/M \to Y$ such that*

$$T = \hat{V}\hat{U}Q_y.$$

Proof. We only need to prove the "only if" part. Let $T : X \to Y$ be a p-compact operator. Then there exists $y = (y_n) \in l_p(Y)$ such that $T = \hat{T}_y \circ Q_y$, that is,

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "Q_y"'] & & Y. \\ & l_{p^*}/ker(T_y) \arrow[ur, "\hat{T}_y"'] & \end{tikzcd}$$

We may assume $y_n \neq 0$ for every n. Let (β_n) be a sequence of positive numbers with $\beta_n \to 0$ such that

$$\sum_n \|y_n\|^p / \beta_n^p < \infty.$$

Let $\lambda = (\lambda_n) = (\|y_n\|/\beta_n)$ and $z_n = y_n/\lambda_n$ for each n. Then $\lambda \in l_p$, $z_n \to 0$ as $n \to \infty$, and

$$T(B_X) \subset \left\{ \sum_n \alpha_n \lambda_n z_n \colon (\alpha_n) \in B_{l_{p^*}} \right\}.$$

Define bounded operators $U : l_{p^*} \to l_1$ and $V : l_1 \to Y$ by

$$U(\alpha) = (\lambda_n \alpha_n), \quad \alpha = (\alpha_n) \in l_{p^*}$$

and

$$V(\gamma) = \sum_n \gamma_n z_n, \quad \gamma = (\gamma_n) \in l_1,$$

respectively. Then it is easy to see that $T_y = VU$, that is,

$$\begin{tikzcd} l_{p^*} \arrow[rr, "T_y"] \arrow[dr, "U"'] & & Y. \\ & l_1 \arrow[ur, "V"'] & \end{tikzcd}$$

Since $(\lambda_n e_n) \in l_p(l_1)$ and $(z_n) \in c_0(Y)$, the operator U is p-compact and V is compact.

Define $\hat{U} : l_{p^*}/ker(T_y) \to l_1/ker(V)$ by

$$\hat{U}\big([\alpha]\big) = \big[(\lambda_n \alpha_n)\big] \in l_1/ker(V)$$

for $\alpha \in l_{p^*}$ and define $\hat{V} : l_1/ker(V) \to Y$ by

$$\hat{V}\big([\gamma]\big) = V(\gamma) \in Y$$

for $\gamma \in l_1$. Clearly, $\hat{U}$ is p-compact, $\hat{V}$ is compact, and

$$\begin{tikzcd} l_{p^*}/ker(T_y) \arrow[rr, "\hat{T}_y"] \arrow[dr, "\hat{U}"'] & & Y. \\ & l_1/ker(V) \arrow[ur, "\hat{V}"'] & \end{tikzcd}$$

Hence $T = \hat{T}_y Q_y = \hat{V}\hat{U} Q_y$. $\square$

4. Denseness of finite rank operators in the topology τ_p

We apply the representation of the continuous linear functionals on $(\mathcal{L}(X, Y), \tau_p)$ and also the factorization of p-compact operators to study the relation of the p-AP with the denseness of finite rank operators in the topology τ_p.

We recall that X has the p-AP is equivalent to that if $f \in (\mathcal{L}(X, X), \tau_p)^*$ and $f(T) = 0$ for all $T \in \mathcal{F}(X, X)$, then $f(id_X) = 0$.

Since every Banach space has the 2-AP [15, Theorem 6.4] (hence the p-AP for every $1 \leqslant p \leqslant 2$), the following results in this section are valid for $2 < p \leqslant \infty$.

Theorem 4.1. *Let $2 < p \leqslant \infty$. The following are equivalent.*

(a) *For every Banach space Y, $\mathcal{K}(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\tau_p}$.*
(b) *For every separable reflexive Banach space Y, $\mathcal{K}(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\tau_p}$.*
(c) *For every Banach space Y and $T \in \mathcal{K}(X,Y)$,*

$$T \in \overline{\{TS\colon S \in \mathcal{F}(X,X)\}}^{\tau_p}.$$

(d) *For every separable reflexive Banach space Y and $T \in \mathcal{K}(X,Y)$,*

$$T \in \overline{\{TS\colon S \in \mathcal{F}(X,X)\}}^{\tau_p}.$$

(e) *X has the p-AP.*

Proof. (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d), and (e) $\Rightarrow$ (a) are trivial. The proof of (b) $\Rightarrow$ (c) follows from that of [10, Theorem (b) $\Rightarrow$ (c)].

(d) $\Rightarrow$ (e) Assume that $f \in (\mathcal{L}(X,X),\tau_p)^*$ and $f(T)=0$ for all $T \in \mathcal{F}(X,X)$. By Theorem 2.5 there exist $(x_n) \in l_p(X)$, $z_j = (\lambda_n^j)_{n=1}^\infty$ in l_{p^*} for each $j \in \mathbb{N}$, (x_j^*) in X^* with $\sum_{j=1}^\infty \|z_j\|_{p^*}\|x_j^*\| < \infty$ and that

$$f(T) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j^*(Tx_n)$$

for every $T \in \mathcal{L}(X,X)$.

In order to complete the proof it is enough to show that

$$f(id_X) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j^*(x_n) = 0.$$

We may assume without loss of generality that $\|x_j^*\| \leqslant 1$ for every j, $x_j^* \to 0$ as $j \to \infty$, and

$$\sum_j \|z_j\|_{p^*} < \infty.$$

By the result of [11] there exists a separable reflexive Banach space Z such that the inclusion mapping $J : Z \to X^*$ is compact and $\overline{abco}(\{x_j^*\}) \subset B_Z$. By the assumption

$$J^* j_X \in \overline{\{J^* j_X S\colon S \in \mathcal{F}(X,X)\}}^{\tau_p} \subset \mathcal{L}\big(X, Z^*\big),$$

where $j_X : X \to X^{**}$ is the canonical isometry. By Theorem 2.5 we can also define $g \in (\mathcal{L}(X,Z^*),\tau_p)^*$ by

$$g(R) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j j_Z\big(x_j^*\big)(Rx_n).$$

An easy computation and the assumption show that for every $S \in \mathcal{F}(X, X)$ we have $g(J^* j_X S) = f(S) = 0$, which implies that

$$0 = g\big(J^* j_X\big) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \lambda_n^j j_Z\big(x_j^*\big)\big(J^* j_X x_n\big) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \lambda_n^j x_j^*(x_n) = f(id_X). \qquad \square$$

The following is the dual version of Theorem 4.1.

Theorem 4.2. *Let $2 < p \leqslant \infty$. The following are equivalent.*

(a) *For every separable reflexive Banach space Y, $\mathcal{K}(Y, X) \subset \overline{\mathcal{F}(Y, X)}^{\tau_p}$.*
(b) *For every Banach space Y, $\mathcal{K}(Y, X) \subset \overline{\mathcal{F}(Y, X)}^{\tau_p}$.*
(c) *X has the p-AP.*

Proof. (c) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b) Let Y be a Banach space and let $T \in \mathcal{K}(Y, X)$, let K be a p-compact set in Y and let $\varepsilon > 0$. By a result of [11] there exist a separable reflexive Banach Z, a compact operator $R : Y \to Z$, and a compact operator $S : Z \to X$ such that $T = SR$. Since $R(K)$ is a p-compact set in Z, by the assumption (a) there exists a $U \in \mathcal{F}(Z, X)$ so that

$$\sup_{y \in K} \|URy - Ty\| = \sup_{y \in K} \|URy - SRy\| \leqslant \varepsilon.$$

Since $UR \in \mathcal{F}(Y, X)$, the conclusion follows.

(b) $\Rightarrow$ (c) By [6, Theorem 2.1] X has the p-AP if and only if for every Banach space Y, $\mathcal{K}_p(Y, X) \subset \overline{\mathcal{F}(Y, X)}^{\|\cdot\|}$. Let $T \in \mathcal{K}_p(Y, X)$ and $\varepsilon > 0$. By Theorem 3.1 there exist a Banach space Z, a p-compact operator $R : Y \to Z$, and a compact operator $S : Z \to X$ such that $T = SR$. By the assumption (b) there exists $F \in \mathcal{F}(Z, X)$ such that

$$\|T - FR\| = \|SR - FR\| = \sup_{z \in R(B_Y)} \|Sz - Fz\| \leqslant \varepsilon.$$

Hence $T \in \overline{\mathcal{F}(Y, X)}^{\|\cdot\|}$. $\square$

For complex Banach spaces X and Y, let $\mathcal{H}_K(X, Y)$ be the space of compact holomorphic mappings from X into Y (see [1]) and let $\mathcal{H}(X)$ be the space of scalar valued holomorphic mappings on X. In [6, Corollary 2.7] it was shown that, if $\mathcal{H}_K(Y, X) \subset \overline{\mathcal{H}(Y) \otimes X}^{\tau}$ for every separable reflexive Banach space Y, then X has the AP. Theorem 4.2 can be applied to show the following.

Corollary 4.3. *Let $2 < p < \infty$. If $\mathcal{H}_K(Y, X) \subset \overline{\mathcal{H}(Y) \otimes X}^{\tau_p}$ for every separable reflexive Banach space Y, then X has the p-AP.*

5. The p-compact approximation property

For $1 \leqslant p \leqslant \infty$, a Banach space X is said to have the *p-compact approximation property* (p-CAP) if $id_X \in \overline{\mathcal{K}(X, X)}^{\tau_p}$. Then the ∞-CAP is well known as the *compact approximation* *property* (CAP) (see [2, Section 8]). In general, the p-CAP (resp. CAP) is weaker than the p-AP (resp. AP). Willis [17] constructed a Banach space having the CAP, which fails to have the AP. We show for every $2 < p < \infty$ that a variant of the Willis space has the CAP, hence p-CAP, but it fails to have the p-AP. Further, the Davie space in c_0 [5] does not have even the p-CAP.

For each $k \geqslant 0$, let G_k be the subset of $\mathbb{N}$ and for each $1 \leqslant j \leqslant 2^k$, let $e_j^k : \mathbb{N} \to \mathbb{C}$ be the function in Section 2. Let E be the closed linear span of $\{e_j^k\colon k \geqslant 0, j = 1, \dots, 2^k\}$ in c_0. For every $k \geqslant 0$ and for every $g \in G_k$, let $\beta^k \in \mathcal{L}(E, E)^*$ and $\phi_g^k \in E$ be defined in the same way as in Section 2. Then it was shown in [5] that

(5.1) there exists an $A > 0$ such that for every $k \geqslant 0$

$$\|\phi_g^k\|_\infty \leqslant A(k+1)^{\frac{1}{2}} 2^{-\frac{k}{2}}$$

for all $g \in G_k$,

(5.2) for every $T \in \mathcal{L}(E, E)$

$$\lim_k \beta^k(T) = \beta(T) \text{ exists}, \qquad \left|\beta(T)\right| \leqslant 3 \sup_n \|T x_n\|_\infty, \quad \text{and} \quad \beta(id_E) = 1,$$

where $(x_n) \subset E$ is the sequence defined in (2.2).

Theorem 5.1. *The Davie space E fails to have the p-CAP for every $2 < p < \infty$.*

Proof. Let $2 < p < \infty$. It follows from (5.1) that

$$\begin{aligned}
\sum_n \|x_n\|_\infty^p &= \|e_1^0\|_\infty^p + \sum_{k=0}^\infty \sum_{g \in G_k} \|(k+1)^2 \phi_g^k\|_\infty^p \\
&\leqslant 1 + A^p \sum_{k=0}^\infty 3 \cdot 2^k (k+1)^{\frac{5}{2}p} 2^{-\frac{pk}{2}} \\
&= 1 + 3A^p \sum_{k=0}^\infty (k+1)^{\frac{5}{2}p} 2^{\frac{(2-p)k}{2}} < \infty,
\end{aligned}$$

hence $(x_n) \in l_p(E)$. We also deduce from (5.2) that $\beta \in (\mathcal{L}(E, E), \tau_p)^*$ and $\beta(id_E) = 1$. It is enough to show that $\beta(T) = 0$ for every $T \in \mathcal{K}(E, E)$, because it implies $id_E \notin \overline{\mathcal{K}(E, E)}^{\tau_p}$. For each $k \geqslant 0$, recall that the linear functional β^k on $\mathcal{L}(E, E)$ is defined by

$$\beta^k(T) = \frac{1}{2^k} \sum_{j=1}^{2^k} \alpha_j^k(T e_j^k)$$

and for each $y \in E$

$$\frac{1}{2^k} \sum_{j=1}^{2^k} \left|\alpha_j^k(y)\right| \to 0$$

as $k \to \infty$ (see [5]). Now let $T \in \mathcal{K}(E, E)$ and $\varepsilon > 0$. Since

$$C = \left\{e_j^k\colon\ k \geqslant 0,\ j = 1, \ldots, 2^k\right\}$$

is bounded in E, $T(C)$ has an ε-net $\{y_i\}_{i=1}^n$. Fix $k \geqslant 0$. For each $1 \leqslant j \leqslant 2^k$ there exists $y_{k_j} \in \{y_i\}_{i=1}^n$ such that $\|Te_j^k - y_{k_j}\|_\infty \leqslant \varepsilon$. Then

$$\begin{aligned}
\left|\beta^k(T)\right| &= \frac{1}{2^k}\left|\sum_{j=1}^{2^k} \alpha_j^k\left(Te_j^k - y_{k_j} + y_{k_j}\right)\right| \\
&\leqslant \frac{1}{2^k}\sum_{j=1}^{2^k}\left|\alpha_j^k(y_{k_j})\right| + \frac{1}{2^k}\sum_{j=1}^{2^k}\left|\alpha_j^k\left(Te_j^k - y_{k_j}\right)\right| \\
&\leqslant \frac{1}{2^k}\sum_{j=1}^{2^k}\sum_{i=1}^{n}\left|\alpha_j^k(y_i)\right| + \varepsilon.
\end{aligned}$$

Hence $|\beta(T)| = \lim_k |\beta^k(T)| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\beta(T) = 0$. $\square$

We now construct a simple variant of the Willis space [17]. Let $2 < p < \infty$ and consider the sequence $(x_n) \subset E$ defined in (2.2). Then id_E cannot be approximated on the set $\{x_n\}_{n=1}^\infty$ by the elements of $\mathcal{K}(E, E)$, because

$$1 = \left|\left|\beta(T)\right| - \left|\beta(id_E)\right|\right| \leqslant \left|\beta(T - id_E)\right| \leqslant 3\sup_n \left\|(T - id_E)(x_n)\right\|_\infty$$

for every $T \in \mathcal{K}(E, E)$. This fact is applied in the proof of Theorem 5.2. Without loss of generality we may assume that $x_n \neq 0$ for every n. Then there exists a decreasing sequence (β_n) such that $\beta_n \to 0$, $0 < \beta_n < 1$, and

$$\sum_n \left(\|x_n\|_\infty / \beta_n\right)^p \leqslant 1.$$

Let $\lambda_n = \|x_n\|_\infty / \beta_n$ and $z_n = x_n/\lambda_n$ for each n. Clearly, $(\lambda_n) \in B_{l_p}$, $1 > \|z_n\|_\infty \geqslant \|z_{n+1}\|_\infty \to_n 0$ with $\|z_n\|_\infty > 0$ for every n.

For $0 \leqslant t < 1$, let

$$U_t = \left\{\sum_n \alpha_n z_n / \|z_n\|_\infty^t\colon\ (\alpha_n) \in B_{l_1}\right\}.$$

We can check that $x_n, z_n \in U_t$ for every $0 \leqslant t < 1$ and $U_s \subset U_t$ for $0 \leqslant s < t < 1$. Let Y_t be the linear span of U_t with the norm

$$\|x\|_t = \inf\{\lambda > 0\colon\ x \in \lambda U_t\}.$$

Then $(Y_t, \|\cdot\|_t)$ is a Banach space with the closed unit ball U_t, and $Y_s \subset Y_t$ and $\|x\|_t \leqslant \|x\|_s$ for $0 \leqslant s < t < 1$. Let $L_t : Y_t \to E$ be the inclusion mapping. Then L_t is a compact operator. Let

$$\mathcal{Z} = \operatorname{span}\{y\chi_{(s,t]}\colon\ 0 \leqslant s < t \leqslant 1,\ y \in Y_s\},$$

where $\chi_{(s,t]}$ is the characteristic function of $(s,t]$. We define a norm on $\mathcal{Z}$ by

$$\|f\| = \int_0^1 \|f(r)\|_r \, dr.$$

Now let Z be the completion of $\mathcal{Z}$ with respect to this norm. Define the operator $R : Y_{\frac{1}{2}} \to Z$ by

$$R(y) = 2y\chi_{(\frac{1}{2},1]}$$

and $J : Z \to E$ by

$$J(f) = \int_0^1 f(r)\, dr.$$

Then $JR = L_{\frac{1}{2}}$ and $\|J\| \leqslant 1$.

Theorem 5.2. *The Banach space Z has the CAP, but it fails to have the p-AP for every $2 < p < \infty$.*

Proof. By the same argument as in [17, Proposition 2] Z has the CAP, in fact, the *metric* CAP. Suppose that Z has the p-AP for some $2 < p < \infty$. Then $R \in \overline{\mathcal{F}(Y_{\frac{1}{2}}, Z)}^{\tau_p}$. Since

$$z_n / \|z_n\|_\infty^{1/2} \in U_{\frac{1}{2}},$$

we can see

$$\|z_n\|_{\frac{1}{2}} \leqslant \|z_n\|_\infty^{1/2},$$

hence $\|z_n\|_{\frac{1}{2}} \to 0$ as $n \to \infty$. In particular,

$$\sum_n \|x_n\|_{\frac{1}{2}}^p = \sum_n \|\lambda_n z_n\|_{\frac{1}{2}}^p < \infty.$$

Therefore $\{x_n\}_{n=1}^\infty$ is a relatively p-compact set in $Y_{\frac{1}{2}}$. Since

$$R \in \overline{\mathcal{F}(Y_{\frac{1}{2}}, Z)}^{\tau_p},$$

given $\varepsilon > 0$, there exists $S \in \mathcal{F}(Y_{\frac{1}{2}}, Z)$ such that

$$\sup_n \|JSx_n - L_{\frac{1}{2}}x_n\|_\infty = \sup_n \|JSx_n - JRx_n\|_\infty \leqslant \sup_n \|Sx_n - Rx_n\|_Z \leqslant \varepsilon.$$

We have shown that $L_{\frac{1}{2}}$ can be approximated on the relatively p-compact set $\{x_n\}_{n=1}^\infty$ in $Y_{\frac{1}{2}}$ by the elements of $\mathcal{F}(Y_{\frac{1}{2}}, E)$. Now let $\varepsilon > 0$ and let $T \in \mathcal{F}(Y_{\frac{1}{2}}, E)$ such that

$$\sup_n \|L_{\frac{1}{2}}x_n - Tx_n\|_\infty \leqslant \frac{\varepsilon}{2}.$$

We may write $T = \sum_{k=1}^N \psi_k(\cdot)u_k$, where $\psi_k \in Y_{\frac{1}{2}}^*$, $u_k \in E$ for each $k = 1, \dots, N$ and $\sum_{k=1}^N \|u_k\|_\infty = 1$. Since $L_{\frac{1}{2}}$ is injective, $Y_{\frac{1}{2}}^* = \overline{L_{\frac{1}{2}}^*(E^*)}^{\tau_p}$. Thus for each $k = 1, \dots, N$ there exists $u_k^* \in E^*$ such that

$$\sup_n \bigl|\psi_k(x_n) - L_{\frac{1}{2}}^*\bigl(u_k^*\bigr)(x_n)\bigr| \leqslant \frac{\varepsilon}{2}.$$

Consider $\sum_{k=1}^N u_k^*(\cdot)u_k \in \mathcal{F}(E, E)$. Then for every n

$$\begin{aligned}\Biggl\|\sum_{k=1}^N u_k^*(x_n)u_k - x_n\Biggr\|_\infty &\leqslant \Biggl\|\sum_{k=1}^N u_k^*(x_n)u_k - Tx_n\Biggr\|_\infty + \|Tx_n - L_{\frac{1}{2}}x_n\|_\infty \\ &\leqslant \sum_{k=1}^N \bigl|u_k^*(x_n) - \psi_k(x_n)\bigr| \|u_k\|_\infty + \frac{\varepsilon}{2} \\ &= \sum_{k=1}^N \bigl|L_{\frac{1}{2}}^*\bigl(u_k^*\bigr)(x_n) - \psi_k(x_n)\bigr| \|u_k\|_\infty + \frac{\varepsilon}{2} \leqslant \varepsilon.\end{aligned}$$

Hence id_E can be approximated on the set $\{x_n\}_{n=1}^\infty$ by the elements of $\mathcal{F}(E, E)$, which is a contradiction. $\square$

Acknowledgment

The authors thank the anonymous referee whose suggestions have improved the final form of the paper.

References

[1] R.M. Aron, M. Schottenloher, Compact holomorphic mappings on Banach spaces and the approximation property, J. Funct. Anal. 21 (1976) 7–30.

[2] P.G. Casazza, Approximation properties, in: W.B. Johnson, J. Lindenstrauss (Eds.), Handbook of the Geometry of Banach Spaces, vol. 1, Elsevier, Amsterdam, 2001, pp. 271–316.

[3] C. Choi, J.M. Kim, Locally convex vector topologies on $\mathcal{B}(X, Y)$, J. Korean Math. Soc. 45 (2008) 1677–1703.

[4] C. Choi, J.M. Kim, K.Y. Lee, Right and left weak approximation properties in Banach spaces, Canad. Math. Bull. 52 (2009) 28–38.

[5] A.M. Davie, The approximation problem for Banach spaces, Bull. London Math. Soc. 5 (1973) 261–266.
[6] J.M. Delgado, E. Oja, C. Piñeiro, E. Serrano, The p-approximation property in terms of density of finite rank operators, J. Math. Anal. Appl. 354 (2009) 159–164.
[7] J. Diestel, J.H. Fourie, J. Swart, The Metric Theory of Tensor Products, Amer. Math. Soc., Providence, 2008.
[8] P. Enflo, A counterexample to the approximation property for Banach spaces, Acta Math. 130 (1973) 309–317.
[9] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc. 16 (1955).
[10] J.M. Kim, New criterion of the approximation property, J. Math. Anal. Appl. 345 (2008) 889–891.
[11] Å. Lima, O. Nygaard, E. Oja, Isometric factorization of weakly compact operators and the approximation property, Israel J. Math. 119 (2000) 325–348.
[12] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I, Sequence Spaces, Springer, Berlin, 1977.
[13] R.E. Megginson, An Introduction to Banach Space Theory, Springer, New York, 1998.
[14] R.A. Ryan, Introduction to Tensor Products of Banach Spaces, Springer, Berlin, 2002.
[15] D.P. Sinha, A.K. Karn, Compact operators whose adjoints factor through subspaces of l_p, Studia Math. 150 (2002) 17–33.
[16] D.P. Sinha, A.K. Karn, Compact operators which factor through subspaces of l_p, Math. Nachr. 281 (2008) 412–423.
[17] G.A. Willis, The compact approximation property does not imply the approximation property, Studia Math. 103 (1992) 99–108.